Exercise 32

Solve the boundary-value problem, if possible.

$$y'' + 4y' + 20y = 0, \quad y(0) = 1, \quad y(\pi) = e^{-2\pi}$$

Solution

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \to \quad y' = re^{rx} \quad \to \quad y'' = r^2 e^{rx}$$

Plug these formulas into the ODE.

$$r^2 e^{rx} + 4(re^{rx}) + 20(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 4r + 20 = 0$$

Solve for r.

$$r = \frac{-4 \pm \sqrt{16 - 4(1)(20)}}{2} = \frac{-4 \pm \sqrt{-64}}{2} = -2 \pm 4i$$

$$r = \{-2 - 4i, -2 + 4i\}$$

Two solutions to the ODE are $e^{(-2-4i)x}$ and $e^{(-2+4i)x}$. By the principle of superposition, then,

$$\begin{aligned} y(x) &= C_1 e^{(-2-4i)x} + C_2 e^{(-2+4i)x} \\ &= C_1 e^{-2x} e^{-4ix} + C_2 e^{-2x} e^{4ix} \\ &= e^{-2x}(C_1 e^{-4ix} + C_2 e^{4ix}) \\ &= e^{-2x}[C_1(\cos 4x - i \sin 4x) + C_2(\cos 4x + i \sin 4x)] \\ &= e^{-2x}[(C_1 + C_2)\cos 4x + (-iC_1 + iC_2)\sin 4x] \\ &= e^{-2x}(C_3 \cos 4x + C_4 \sin 4x). \end{aligned}$$

Apply the boundary conditions to determine C_3 and C_4.

$$y(0) = C_3 = 1$$

$$y(\pi) = e^{-2\pi} C_3 = e^{-2\pi}$$

Solving this system of equations yields $C_3 = 1$. Therefore, the solution to the boundary value problem is

$$y(x) = e^{-2x}(\cos 4x + C_4 \sin 4x),$$

where C_4 remains arbitrary.